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Convergence Rates of Regularized Approximation Processes[1]

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We define the concept of an *A-regularized approximation process* and prove for it uniform convergence theorems and strong convergence theorems with optimal and non-optimal rates. The sharpness of non-optimal convergence is also established. The general results provide a unified approach to dealing with convergence rates of various approximation processes, and also of local ergodic limits as well. As applications, approximation theorems, and local Abelian and Cesáro ergodic theorems with rates are deduced for n-times integrated solution families for Volterra integral equations, which include n-times integrated semigroups and cosine functions as special cases. Applications to (Y)-semigroups and tensor product semigroups are also discussed. © 2002 Elsevier Science (USA)

Key Words: regularized approximation process; saturation property; non-optimal convergence; K-functional; Grothendieck space; the Dunford–Pettis property; generalized Hille–Yosida operator; local Abelian and Cesáro ergodic theorems; n-times integrated solution family; n-times integrated semigroup; n-times integrated cosine function; tensor product semigroup.

1. INTRODUCTION

A net $\{T_\alpha\}_{\alpha \in \Gamma}$ of bounded linear operators on a Banach space X is called an *approximation process* on X if $\|T_\alpha x - x\| \to 0$ for all $x \in X$. The process $\{T_\alpha\}$ is said to possess the *saturation property* if there exists a positive function $e(\alpha)$ tending to 0 such that every $x \in X$ for which $\|T_\alpha x - x\| = o(e(\alpha))$ is an invariant element of $\{T_\alpha\}$, i.e., $T_\alpha x = x$ for all α, and if the set

$$F[X; T_\alpha] = \{x \in X; \|T_\alpha x - x\| = O(e(\alpha))\}$$

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contains at least one non-invariant element. In this case, the approximation process $\{T_\alpha\}$ is said to have *optimal approximation order* $O(e(\alpha))$ or to be *saturated in* X *with order* $O(e(\alpha))$, and $F[X; T_\alpha]$ is called its *Favard class* or *saturation class*. See, e.g., [4, p. 434] for the above definition.

Thus the saturation concept consists of determination of the optimal approximation order $O(e(\alpha))$ and the corresponding saturation class. In [4], Butzer and Nessel considered saturation with the optimal approximation order $O(\rho^{-\theta})(\rho \to \infty)$ (i.e., $e(\rho) = \rho^{-\theta}$, $\rho > 0$) for $\theta > 0$ for approximation processes which satisfy some extra conditions. In [9], under some further assumptions, the non-optimal rate $O(\rho^{-\gamma})(\rho \to \infty)$, $0 < \gamma < \theta$, is described in terms of that of a K-functional, and is shown to be sharp. These general results provide a unified approach to convergence rates for various approximation processes, such as n-times integrated semigroups and n-times integrated cosine functions [9], in particular.

However, from the viewpoint of application, the assumptions required in [4, 9] are rather complicated. The aim of this paper is to formulate an easy-implementing format for treating saturation and non-optimal rates for processes under simpler assumptions and with the more general optimal approximation order $O(e(\alpha))$. For the A-regularized approximation process to be defined in Section 2, theorems about saturation (Theorem 2.6), non-optimal convergence (Theorem 2.7) and the sharpness of non-optimal convergence rate (Theorem 2.9) will be established. We also show in Theorem 2.4 that an A-regularized approximation process on a Grothendieck space with the Dunford–Pettis property converges uniformly if its range is contained in the domain of A.

To demonstrate the usefulness of our general results, we consider their applications to some examples. In Section 3, we deduce a local Abelian ergodic theorem (Theorem 3.4) with rates for generalized Hille–Yosida operators, which generalizes results in [5, 6] on C_0-semigroups and cosine operator functions. In Section 4, approximation at 0 of an n-times integrated solution family $S(\cdot)$ will be considered. In attempt to obtain wider result, we shall consider some kind of local means $Q_m(t)$, $t \geqslant 0$, $m \geqslant 0$ of $(n!/t^n)\, S(t)$, instead of $S(t)$ itself. These local means include in particular $Q_0(t) = (n!/t^n)\, S(t)$, and $Q_1(t) = (a * S(t))/(a * j_n(t))$. Thus we can deduce approximation theorem and local ergodic theorems with rates (Theorems 4.4, 4.5, and 4.7) for n-times integrated solution families. In particular, they reduce to approximation and local Cesáro and Abelian ergodic theorems (Theorems 4.8 and 4.9) for n-times integrated semigroups and n-times integrated cosine functions (cf. [9]), which contain as special cases some results on C_0-semigroups [7] and cosine operator functions [5]. In Section 5, we will deduce approximation theorems for (Y)-semigroups and tensor product semigroups. Finally, we remark that the general results in [4, 9] seem not applicable to the derivation of our results in Sections 4 and 5.

2. REGULARIZED APPROXIMATION PROCESSES

This section is devoted to general results on the strong and uniform convergence of regularized approximation processes, with emphasis on their optimal and non-optimal convergence rates.

We start with the following definition of a regularized approximation process. In the sequel, we use the notations $D(T)$, $R(T)$, and $N(T)$, for the domain, range, and null space, respectively, of a linear operator T.

DEFINITION 2.1. Let $e(\alpha)$ be a positive function tending to 0. A net $\{T_\alpha\}$ of bounded linear operators on X is called an *A-regularized approximation process of order $O(e(\alpha))$* on X if it is uniformly bounded, i.e., $\|T_\alpha\| \leqslant M$ for some $M > 0$ and all α, and satisfies

(A1) there are a (necessarily densely defined) closed linear operator A on X and a uniformly bounded approximation process $\{S_\alpha\}$ on X such that

$$R(S_\alpha) \subset D(A) \qquad \text{and} \qquad S_\alpha A \subset AS_\alpha = (e(\alpha))^{-1}(T_\alpha - I) \qquad \text{for all } \alpha.$$

In this case, the process $\{S_\alpha\}$ is called a *regularization process* associated with $\{T_\alpha\}$.

Notice that when $e(\rho) = \rho^{-\theta}$, $\rho > 0$, the above definition reduces to the one that was considered in [3, 4]. In the following, $\{T_\alpha\}$ denotes an A-regularized approximation process of order $O(e(\alpha))$ with regularization process $\{S_\alpha\}$.

LEMMA 2.2. (i) *$x \in D(A)$ and $y = Ax$ if and only if $y = \lim_\alpha (e(\alpha))^{-1}(T_\alpha - I)\, x$.*

(ii) *$D(A)$ is dense in X, and $\|T_\alpha x - x\| \to 0$ for all $x \in X$.*

(iii) *If A is bounded, then $\|T_\alpha - I\| = O(e(\alpha)) \to 0$.*

(iv) *$\|T_\alpha - I\| \to 0$ implies $A \in B(X)$ if either $R(T_\alpha) \subset D(A)$ for all α, or S_α and T_α satisfy the following condition:*

(A2) *$\|T_\alpha - I\| \to 0$ implies $\|S_\alpha - I\| \to 0$.*

Proof. (i) This follows easily from (A1) and the closeness of A.

(ii) For any $x \in X$ we have $x = \lim S_\alpha x \in \overline{D(A)}$. Hence $X = \overline{D(A)}$. Since $\|T_\alpha x - x\| = e(\alpha)\, \|S_\alpha Ax\| \to 0$ for all $x \in D(A)$, the convergence for all $x \in X$ follows from the uniform boundedness of $\{T_\alpha\}$.

(iii) This follows from (A1) and the boundedness of A.

(iv) Suppose $\|T_\alpha - I\| \to 0$. Then T_α is invertible for some α. If $R(T_\alpha) \subset D(A)$, then $X = R(T_\alpha) \subset D(A)$. If (A2) holds, then by (A2) we have

$$\|S_\alpha T_\alpha - I\| \leqslant \|S_\alpha T_\alpha - S_\alpha\| + \|S_\alpha - I\| \leqslant M_1\, \|T_\alpha - I\| + \|S_\alpha - I\| \to 0.$$

Hence $S_{\alpha_1}T_{\alpha_1}$ is invertible for some α_1, so that $D(A) \supset R(S_{\alpha_1}) \supset R(S_{\alpha_1}T_{\alpha_1}) = X$. In both case we have $D(A) = X$ and so A is bounded on X.

PROPOSITION 2.3. (i) *$\{T_\alpha^* x^*\}$ converges weakly* to x^* for all $x^* \in X^*$.*

(ii) *Under the additional condition that $R(T_\alpha) \subset D(A)$ for all α, the following are equivalent:* (a) $\|T_\alpha^* x^* - x^*\| \to 0$; (b) *$\{T_\alpha^* x^*\}$ has a weak cluster point;* (c) $x^* \in \overline{D(A^*)}$.

Proof. (i) This follows immediately from (ii) of Lemma 2.2.

(ii) While "(a) ⇒ (b)" is obvious, we show other implications. First, we see that $T_\alpha D(A) \subset D(A)$ and $T_\alpha Ax = AT_\alpha x$ for all $x \in D(A)$. Indeed, if $x \in D(A)$, then by (A1) we have $T_\alpha x = e(\alpha)\, S_\alpha Ax + x \in D(A)$ and $AT_\alpha x = e(\alpha)\, AS_\alpha Ax + Ax = (T_\alpha - I)\, Ax + Ax = T_\alpha Ax$, and under the additional condition $R(T_\alpha) \subset D(A)$, one has $T_\alpha A \subset AT_\alpha$. Then, by the argument in [27, p. 408], it can be shown that $R(T_\alpha^*) \subset D(A^*)$ and $T_\alpha^* A^* \subset A^* T_\alpha^*$. If a subnet $\{T_\beta^* x^*\}$ has a weak limit, then by (i) we have $x^* = \text{w}^*\text{-lim}\, T_\alpha^* x^* = \text{w-lim}\, T_\beta^* x^* \in \overline{D(A^*)}$. This shows "(b) ⇒ (c)."

Similarly, (A1) implies that

$$R(S_\alpha^*) \subset D(A^*) \quad \text{and} \quad S_\alpha^* A^* \subset A^* S_\alpha^* = (e(\alpha))^{-1}\, (T_\alpha^* - I^*) \quad \text{for all } \alpha. \tag{2.1}$$

Since $\{S_\alpha\}$ is uniformly bounded, it follows from (2.1) that $\|T_\alpha^* x^* - x^*\| \to 0$ for all $x^* \in D(A^*)$. Hence (c) implies (a), by the uniform boundedness of $\{T_\alpha\}$.

A Banach space X is called a *Grothendieck space* if every weakly* convergent sequence in X^* is weakly convergent (see, e.g., [28] for equivalent definitions), and is said to have the *Dunford–Pettis property* if $\langle x_n, x_n^* \rangle \to 0$ whenever $x_n \to 0$ weakly in X and $x_n^* \to 0$ weakly in X^*. The spaces L^∞, H^∞, and $B(S, \Sigma)$ are particular examples of Grothendieck spaces with the Dunford–Pettis property (see [19]). A common phenomenon in such spaces is that strong operator convergence often implies uniform operator convergence. For instance, a theorem of Coulhon [10] asserts that if an approximation process $\{T_n\}$ on a Grothendieck space X with the Dunford–Pettis property is uniformly power bounded, i.e., $\|T_n^k\| \leqslant M$ for all $n, k \geqslant 1$, and if the dual operators $\{T_n^*\}$ is an approximation process on X^*, then $\|T_n - I\| \to 0$. The following is a theorem of this type for regularized approximation processes.

THEOREM 2.4. *Let $\{T_\alpha\}$ be an A-regularized approximation process of order $O(e(\alpha))$ on a Grothendieck space X with the Dunford–Pettis property. If $R(T_\alpha) \subset D(A)$ for all α, then $A \in B(X)$ and $\|T_\alpha - I\| = O(e(\alpha))$.*

Proof. Take a subsequence $\{T_{\alpha_n}\}$ and let $V_n = T_{\alpha_n} - I$. Since $T_\alpha - I$ converges to 0 strongly on X, $V_n^* x_n^*$ converges weakly* and hence weakly to 0 for any bounded sequence $\{x_n^*\}$ in X^*. In particular, $\{V_n^* x^*\}$ has the weak limit 0, and hence x^* is a weak cluster point of $\{T_\alpha^* x^*\}$ for every $x^* \in X^*$, so that, by Proposition 2.3(ii), $\|T_\alpha^* x^* - x^*\| \to 0$ for all $x^* \in X^*$. It follows that $V_n x_n$ converges weakly to 0 for any bounded sequence $\{x_n\}$ in X. Since X has the Dunford–Pettis property, we have $\langle V_n^2 x_n, x_n^* \rangle = \langle V_n x_n, V_n^* x_n^* \rangle \to 0$. Since we can choose unit x_n and x_n^* such that $\|V_n^2\| \leqslant \langle V_n^2 x_n, x_n^* \rangle + 1/n$, it follows that $\|V_n^2\| \to 0$, so that for large n, $V_n^2 - I$ is invertible, and so is $V_n + I = T_{\alpha_n}$. Hence $X = R(T_{\alpha_n}) \subset D(A)$, and A is bounded. Now Lemma 2.2(iii) implies that $\|T_\alpha - I\| = O(e(\alpha))$.

As usual the rates of convergence will be characterized by means of *K-functional* and *relative completion*, which we recall now.

DEFINITION 2.5. Let X be a Banach space with norm $\|\cdot\|_X$, and Y a submanifold with seminorm $\|\cdot\|_Y$. The *K-functional* is defined by

$$K(t, x) := K(t, x, X, Y, \|\cdot\|_Y) = \inf_{y \in Y} \{\|x - y\|_X + t\,\|y\|_Y\}.$$

If Y is also a Banach space with norm $\|\cdot\|_Y$, then the *completion of Y relative to X* is defined as

$$\tilde{Y}^X := \{x \in X : \exists \{x_n\} \subset Y \text{ such that } \lim_{n \to \infty} \|x_n - x\|_X = 0 \text{ and } \sup_n \|x_n\|_Y < \infty\}.$$

It is known [3] that $K(t, x)$ is a bounded, continuous, monotone increasing and subadditive function of t for each $x \in X$, and $K(t, x, X, Y, \|\cdot\|_Y) = O(t)$ if and only if $x \in \tilde{Y}^X$. With these terminologies we now prove theorems for convergence rates. The following is an optimal convergence (saturation) theorem.

THEOREM 2.6. *Let $\{T_\alpha\}$ be an A-regularized approximation process of order $O(e(\alpha))$, and let $D(A)$ be equipped with the graph norm $\|\cdot\|_{D(A)}$. For $x \in X$, we have:*

(i) $\|T_\alpha x - x\| = o(e(\alpha))$ *if and only if* $x \in N(A)$, *if and only if* $T_\alpha x = x$ *for all* α.

(ii) *The following are equivalent:*

(a) $\|T_\alpha x - x\| = O(e(\alpha))$;

(b) $x \in \widetilde{D(A)}^X$;

(c) $x \in D(A)$ *in the case that X is reflexive.*

Proof. (i) This is an immediate consequence of (A1).

(ii) (a) $\Rightarrow$ (b). By the assumption (A1), there is an $M_1 > 0$ such that $\|S_\alpha\| \leqslant M_1$ for all α. If $\|T_\alpha x - x\| = O(e(\alpha))$, then $\|T_\alpha x - x\| \leqslant M_3 e(\alpha)$ for some $M_3 > 0$, and so

$$\|S_\alpha x\|_{D(A)} = \|S_\alpha x\| + \|AS_\alpha x\| = \|S_\alpha x\| + \left\| \frac{1}{e(\alpha)} (T_\alpha x - x) \right\| \leqslant M_1 \|x\| + M_3.$$

Let $\{x_n := S_{\alpha_n} x\}$ be any subsequence of $\{S_\alpha x\}$. Then $\{x_n\} \subset D(A)$ and $x_n \to x$, by (A1). Hence we have $x \in \widetilde{D(A)}^X$.

(b) $\Rightarrow$ (a) and (c). Assumption (A1) implies

$$\frac{1}{e(\alpha)} \|T_\alpha y - y\| \leqslant M_1 \|Ay\| \leqslant M_1 \|y\|_{D(A)} \qquad \text{for all} \qquad y \in D(A) \qquad \text{and } \alpha. \tag{2.2}$$

If $x \in \widetilde{D(A)}^X$, then there is a sequence $\{x_n\} \subset D(A)$ such that $\|x_n\|_{D(A)} \leqslant M_2$ for some $M_2 > 0$ and $\|x_n - x\| \to 0$. Substituting $y = x_n$ into (2.2) and then letting $n \to \infty$, we obtain $\|T_\alpha x - x\| \leqslant M_1 M_2 e(\alpha)$.

If X is reflexive, since $\|Ax_n\| \leqslant M_2$, by the Alaoglu theorem there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Ax_{n_k} \to y$ weakly. Then the closedness of A implies that A is weakly closed, thus $x \in D(A)$ and $y = Ax$ (see [11, Problem 1.4]) and $\widetilde{D(A)}^X \subset D(A)$. Hence, when X is reflexive, one has $\widetilde{D(A)}^X = D(A)$ and (b) becomes (c).

The next theorem is about non-optimal convergence.

THEOREM 2.7. *Let $0 \leqslant e(\alpha) \leqslant f(\alpha) \to 0$. If $K(e(\alpha), x, X, D(A), \|\cdot\|_{D(A)}) = O(f(\alpha))$, then $\|T_\alpha x - x\| = O(f(\alpha))$. The converse statement is also true under the following assumption:*

(A3) $\|S_\alpha x - x\| = O(f(\alpha))$ *whenever* $\|T_\alpha x - x\| = O(f(\alpha))$.

Proof. By (A1) and (2.2) we have for any $y \in D(A)$ and α

$$\begin{aligned} \|T_\alpha x - x\| &\leqslant \|(T_\alpha - I)(x - y)\| + \|T_\alpha y - y\| \\ &\leqslant (M+1) \|x - y\| + e(\alpha) M_1 \|y\|_{D(A)} \\ &\leqslant \max\{M+1, M_1\}(\|x - y\| + e(\alpha) \|y\|_{D(A)}), \end{aligned}$$

and by taking infimum over all $y \in D(A)$, we arrive at

$$\|T_\alpha x - x\| \leqslant \max\{M+1, M_1\} K(e(\alpha), x, X, D(A), \|\cdot\|).$$

For the converse, suppose $\|T_\alpha x - x\| = O(f(\alpha))$. Then by (A1) and (A3) we have for all α

$$\begin{aligned} K(e(\alpha), x, X, D(A), \|\cdot\|_{D(A)}) &\leqslant \|x - S_\alpha x\| + e(\alpha)\, \|S_\alpha x\|_{D(A)} \\ &\leqslant \|x - S_\alpha x\| + \|T_\alpha x - x\| + e(\alpha)\, M_1\, \|x\| \\ &= O(f(\alpha)). \end{aligned}$$

To consider the sharpness of approximation, we need a result of Davydov [12, Theorem 1]. For easier application to Theorem 2.9 we formulate it as the following form.

Proposition 2.8. *Let $\{p_\alpha\}$ be a net of continuous seminorms on a Banach space X satisfying the conditions:*

(a) $\overline{\lim}_\alpha \|p_\alpha\| = \infty$, *where* $\|p_\alpha\| := \sup\{p_\alpha(x);\, x \in X,\, \|x\| \leqslant 1\}$;

(b) *the set* $\{x \in X;\, \lim_\alpha p_\alpha(x) = 0\}$ *is dense in* X.

Then there exists an element $x_0 \in X$ such that $\sup_\alpha p_\alpha(x_0) \leqslant 1$ and $\overline{\lim}_\alpha p_\alpha(x_0) = 1$.

Theorem 2.9. *Suppose an A-regularized approximation process $\{T_\alpha\}$ and its regularization process $\{S_\alpha\}$ satisfy condition* (A2). *Then A is unbounded if and only if for each $f(\alpha)$ with $0 \leqslant e(\alpha) < f(\alpha) \to 0$ and $f(\alpha)/e(\alpha) \to \infty$ there exists $x_f \in X$ such that*

$$\|T_\alpha x_f - x_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases}$$

Proof. If A is bounded, then by Lemma 2.2(iii) we have $\|T_\alpha - I\| = O(e(\alpha))$ so that $\|T_\alpha - I\| = o(f(\alpha))$. This shows the sufficiency.

For the necessity, suppose A is unbounded and define $p_\alpha(x) = (f(\alpha))^{-1}\, \|T_\alpha x - x\|$, $x \in X$. Note that p_α is a seminorm on X with $\|p_\alpha\| \leqslant (M+1)/f(\alpha)$. We show that $\{p_\alpha\}$ satisfies the hypothesis of Proposition 2.8.

By Lemma 2.2(iii), we have $\overline{\lim}_\alpha \|T_\alpha - I\| > 0$, so that $\overline{\lim}_\alpha \|p_\alpha\| = \overline{\lim}_\alpha (f(\alpha))^{-1}\, \|T_\alpha - I\| = \infty$. Moreover, we have $p_\alpha(x) = (f(\alpha))^{-1}\, e(\alpha)\, \|S_\alpha A x\| \to 0$ for all $x \in D(A)$, by (A1) and the assumption $f(\alpha)/e(\alpha) \to \infty$. Hence the set $\{x \in X : \lim_\alpha p_\alpha(x) = 0\}$ contains $D(A)$, which is dense in X by (A1).

The hypothesis of Proposition 2.8 being satisfied, it follows that there exists an $x_f \in X$ such that $\sup_\alpha p_\alpha(x_f) \leqslant 1$ and $\overline{\lim}_\alpha p_\alpha(x_f) = 1$; i.e., x_f satisfies $\|T_\alpha x_f - x_f\| = O(f(\alpha))$ and $\|T_\alpha x_f - x_f\| \neq o(f(\alpha))$.

3. LOCAL ABELIAN ERGODIC THEOREMS WITH RATES

Most of the results in this section are extensions of some known results in [5, 6, 15, 16, 20]. We demonstrate them here in order to show how they can be deduced from our general results in Section 2, and to prepare for their applications in Sections 4 and 5.

Let A be a closed linear operator on X such that

$$(3.1)\qquad (\omega, \infty) \subset \rho(A) \quad \text{and} \quad \|\lambda(\lambda - A)^{-1}\| \leqslant M \quad \text{for all} \quad \lambda > \omega.$$

Such A is called a *generalized Hille–Yosida operator of type* (M, ω) [20]. We have that

$$(3.2)\qquad R((\lambda - A)^{-1}) \subset D(A) \quad \text{and}$$
$$(\lambda - A)^{-1} A \subset A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I.$$

Let A_1 be the part of A in $X_1 := \overline{D(A)}$.

THEOREM 3.1. *For a generalized Hille–Yosida operator A we have:*

(i) $\|\lambda(\lambda - A)^{-1} x - x\| \to 0$ as $\lambda \to \infty$ if and only if there is a sequence $\{\lambda_n\} \to \infty$ such that $\{\lambda_n(\lambda_n - A)^{-1} x\}$ converges weakly, if and only if $x \in X_1$.

(ii) $\|\lambda(\lambda - A_1^*)^{-1} x^* - x^*\| \to 0$ as $\lambda \to \infty$ if and only if there is a sequence $\{\lambda_n\} \to \infty$ such that $\{\lambda_n(\lambda_n - A_1^*)^{-1} x^*\}$ converges weakly in X_1^*, if and only if $x^* \in \overline{D(A_1^*)}$.

Proof. (i) From (3.1) and (3.2) one can easily see (e.g., [31, p. 218]) that $\lambda(\lambda - A)^{-1} x \to x$ as $\lambda \to \infty$ if and only if there is a sequence $\{\lambda_n\} \to \infty$ such that $\{\lambda_n(\lambda_n - A)^{-1} x\}$ converges weakly, if and only if $x \in X_1$.

(ii) Since $A(\lambda - A)^{-1} x = \lambda(\lambda - A)^{-1} x - x \in X_1$ for all $x \in X_1$, we see that $(\lambda - A)^{-1} X_1 \subset D(A_1)$ and $(\lambda - A)^{-1} A_1 x = A_1(\lambda - A)^{-1} x = \lambda(\lambda - A)^{-1} x - x$ for all $x \in X_1$ and $\lambda > \omega$, from which it follows that $(\lambda - A_1)(\lambda - A)^{-1} x = x$ for all $x \in X_1$, so that $(\lambda - A_1)^{-1} = (\lambda - A)^{-1}|_{X_1}$. Thus we have

$$(3.3)\qquad R((\lambda - A_1)^{-1}) \subset D(A_1) \quad \text{and}$$
$$(\lambda - A_1)^{-1} A_1 \subset A_1(\lambda - A_1)^{-1} = \lambda(\lambda - A_1)^{-1} - I|_{X_1},$$

and $\lambda(\lambda - A_1)^{-1} x \to x$ for all $x \in X_1$ as $\lambda \to \infty$.

Let $T_\lambda = S_\lambda = \lambda(\lambda - A_1)^{-1}$ for $\lambda > \omega$. The above argument shows that $\{T_\lambda\}$ is an A_1-regularized approximation process of order $O(\lambda^{-1})(\lambda \to \infty)$ on X_1, satisfying $R(T_\lambda) \subset D(A_1)$ and having itself as a regularization process (and so (A2) and (A3) hold automatically). This implies A_1 is densely defined in X_1, in particular. Then, from Proposition 2.3 we deduce the assertion (ii).

THEOREM 3.2. *Let A be a generalized Hille–Yosida operator on X.*

(i) $\|\lambda(\lambda-A)^{-1}-I\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $A \in B(X)$. *In this case,* $\|\lambda(\lambda-A)^{-1}-I\| = O(\lambda^{-1})(\lambda \to \infty)$.

(ii) *If* X_1 *is a Grothendieck space with the Dunford–Pettis property,* A *must be bounded on* X.

Proof. Note that $A \in B(X)$ if and only if A_1 is bounded. Indeed, if A_1 is bounded, then by (3.1) and (3.3) we have $\|\lambda(\lambda-A_1)^{-1}-I|_{X_1}\| \leqslant \|A_1\| \, \|(\lambda-A_1)^{-1}\| \to 0$ as $\lambda \to \infty$, so that for large λ, $\lambda(\lambda-A_1)^{-1}$ is invertible in X_1 and so $X_1 \subset (\lambda-A_1)^{-1} X_1 = D(A_1) \subset D(A)$. It follows that $D(A)$ is closed and hence A, as a closed operator, must be bounded. Then, by (3.2), we have $\|\lambda(\lambda-A)^{-1}-I\| \leqslant \|A\| \, \|(\lambda-A)^{-1}\| \to 0$ as $\lambda \to \infty$, so that $D(A)=X$ and $A \in B(X)$. The theorem follows by using this fact and applying Lemma 2.2 and Theorem 2.4 to $T_\lambda = S_\lambda = \lambda(\lambda-A)^{-1}|_{X_1}$.

Remarks. Part (ii) of Theorem 3.2 was proved in [19] for generators of C_0-semigroups and in [25, 26] for generators of cosine operator functions. For more general results than Theorems 3.1 and 3.2 for pseudo-resolvents we refer to [26, Theorems 1 and 6].

To deduce convergence theorem with rates, we first recall the following lemma, a proof of which can be found in [20, Lemma 3.3.2].

LEMMA 3.3. *Let A be a generalized Hille–Yosida operator of type (M, ω), and let A_1 be its part in $X_1 = \overline{D(A)}$. Then for $x \in X_1$ we have*

$$\begin{aligned} K(t, x, X, D(A), \|\cdot\|_{D(A)}) &\leqslant K(t, x, X_1, D(A_1), \|\cdot\|_{D(A_1)}) \\ &\leqslant MK(t, x, X, D(A), \|\cdot\|_{D(A)}). \end{aligned}$$

Thus we deduce from Theorems 2.6, 2.7, 2.9, and Lemma 3.3 the following theorem.

THEOREM 3.4. *Let A be generalized Hille–Yosida operator, and let A_1 be the part of A in $X_1 = \overline{D(A)}$. Then the following are true for $x \in X_1$ and $0 < \beta \leqslant 1$:*

(i) $\|\lambda(\lambda-A)^{-1} x - x\| = o(\lambda^{-1})(\lambda \to \infty)$ *if and only if* $x \in N(A)$.

(ii) $\|\lambda(\lambda-A)^{-1} x - x\| = O(\lambda^{-\beta})(\lambda \to \infty)$ *if and only if* $K(t, x, X, D(A), \|\cdot\|_{D(A)}) = O(t^\beta)(t \to 0^+)$, *if and only if* $x \in \widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta = 1$, *if and only if* $x \in D(A_1)$ *in the case that* $\beta = 1$ *and* X *is reflexive.*

(iii) *A is unbounded if and only if for each $0<\beta<1$ there exists $x^*_\beta \in X_1$ such that*

$$\|\lambda(\lambda-A)^{-1}x^*_\beta - x^*_\beta\| \begin{cases} = O(\lambda^{-\beta}) \\ \neq o(\lambda^{-\beta}) \end{cases} \qquad (\lambda\to\infty).$$

Remark. For work dealing with (ii) of Theorem 3.4 see [16; 20, p. 61], and for parts of the theorem for the cases of generators of C_0-semigroups and of cosine operator functions see [5, 6].

We end this section with an application to the infinitesimal generator A of a semigroup $\{T(t); t>0\}$ of class $(0, A)$ (see [15, pp. 342–344]). By definition, $T(\cdot)$ satisfies: (i) $T(\cdot)$ is strongly continuous on $(0,\infty)$; (ii) $\{T(t)\,x; x\in X, t>0\}$ is dense in X; (iii) $\int_0^1 \|T(t)\,x\|\,dt<\infty$ for each $x\in X$; (iv) $\lim_{\lambda\to\infty}\lambda R(\lambda)\,x=x$ for all $x\in X$, where $R(\lambda)\,x:=\int_0^\infty e^{-\lambda t}T(t)\,x\,dt$ for all $x\in X$ and $\lambda>w$ (the type of $T(\cdot)$).

The operator A_0, defined by $A_0x:=\lim_{t\to 0^+} t^{-1}(T(t)-I)\,x$, is closable. The closure A of A_0 is called the infinitesimal generator of $T(\cdot)$. It is known that if $\lambda>w$, then $\lambda\in\rho(A)$ and $R(\lambda)=(\lambda-A)^{-1}$. From this and (iv) we see that A is a generalized Hille–Yosida operator and is densely defined. Thus, for the infinitesimal generator A of a semigroup of class $(0, A)$, Theorems 3.1, 3.2, and 3.4 hold with $X_1=X$ and $A_1=A$.

4. APPROXIMATION OF N-TIMES INTEGRATED SOLUTION FAMILIES

Let A be a closed linear operator in X and $a\in L^1_{loc}(\mathbb{R}_+)$ be a nondecreasing positive kernel. Consider the Volterra equation,

$$(VE; A, a, f) \qquad u(t)=f(t)+\int_0^t a(t-s)\,Au(s)\,ds, \qquad t\geqslant 0,$$

for $f\in C([0,\infty); X)$.

A family $\{S(t); t\geqslant 0\}$ in $B(X)$ is called an *n-times integrated solution family* for (VE, A, a, f) (see [2, 21]) if

(S1) $S(\cdot)$ is strongly continuous on $[0,\infty)$ and $S(0)=I$ if $n=0$ and 0 if $n\geqslant 1$;

(S2) $S(t)\,x\in D(A)$ and $AS(t)\,x=S(t)\,Ax$ for all $x\in D(A)$ and $t\geqslant 0$;

(S3) $a*S(t)\,x\in D(A)$ and $S(t)\,x=(t^n/n!)\,x+A\int_0^t a(t-s)\,S(s)\,x\,ds$ for all $x\in X$ and $t\geqslant 0$.

A 0-times integrated solution family is also called a *solution family* or *resolvent family* [13, 18, 22].

The notion of an n-times integrated solution family is an extension of the concepts of *n-times integrated semigroups* [1, 17] and *n-times integrated cosine functions* [29] (corresponding to the cases $a \equiv 1$ and $a(t) = t$, respectively). The existence of an n-times integrated solution family for $(VE; A, a, f)$ is equivalent to the existence of unique solution of the Volterra equation $(VE; A, a, (t^n/n!)\, x)$ for every $x \in X$ (see [21, Theorem 2.5]).

We shall assume that

$$\|S(t)\| \leqslant Mt^n \qquad \text{for all} \qquad t \geqslant 0. \tag{4.1}$$

Put $j_n(t) = t^n/n!$ for $t \geqslant 0$ and $n \geqslant 0$ and denote by a_0 the *Dirac measure* δ_0 at 0. For each $m \geqslant 0$, let $a_{m+1}(t) = a * a_m(t)$ for $t \geqslant 0$, and

$$\begin{cases} k_m(0) = 0; \\ k_m(t) = \dfrac{a_{m+1} * j_n(t)}{a_m * j_n(t)} \qquad \text{for} \quad t > 0. \end{cases}$$

We define the local means of $(n!/t^n)\, S(t)$:

$$Q_m(t)\, x = \frac{a_m * S(t)\, x}{a_m * j_n(t)} \qquad \text{for} \quad x \in X \qquad \text{and} \qquad t > 0.$$

In particular, $k_0(t) = (a * j_n(t))/j_n(t)$, $k_1(t) = (a * a * j_n(t))/(a * j_n(t))$, $Q_0(t) = (n!/t^n)\, S(t)$, and $Q_1(t) = (a * S(t))/(a * j_n(t))$, which are $\int_0^t a(s)\, ds$, $(a * a * 1(t))/(a * 1(t))$, $S(t)$, and $(a * S(t))/(a * 1(t))$, respectively, when $n = 0$. Since a is nondecreasing and positive, $a_m(t)$ and $a_m * j_n(t)$ are nondecreasing positive functions of t. Therefore

$$k_m(t) = \frac{1}{a_m * j_n(t)} \int_0^t a(t-s)(a_m * j_n)(s)\, ds \leqslant \int_0^t a(s)\, ds \to 0 \tag{4.2}$$

as $t \to 0$. Note further that

$$\begin{aligned} \|Q_m(t)\, x\| &\leqslant \frac{1}{a_m * j_n(t)} \int_0^t a_m(t-s)\, \|S(s)\, x\|\, ds \\ &\leqslant \frac{M\, \|x\|}{a_m * j_n(t)} \int_0^t a_m(t-s)\, s^n\, ds = Mn!\, \|x\| \end{aligned}$$

and we have

$$\|Q_m(t)\| \leqslant Mn! \qquad (m \geqslant 0,\ t > 0). \tag{4.3}$$

LEMMA 4.1. *Let $S(\cdot)$ be an n-times integrated solution family for (VE, A, a, f) such that $\|S(t)\| \leqslant Mt^n$ for all $t \geqslant 0$, and let A_1 be the part of A in $X_1 := \overline{D(A)}$. Then*

$$(4.4)\qquad Q_0(t)\, D(A) \subset D(A) \quad \textit{and} \quad Q_0(t)\, Ax = AQ_0(t)\, x \quad \textit{for} \quad x \in D(A),$$

$$(4.5)\qquad Q_{m+1}(t)\, X \subset D(A) \quad \textit{and} \quad Q_{m+1}(t)\, A \subset AQ_{m+1}(t) = \frac{1}{k_m(t)}(Q_m(t) - I),$$

$$(4.6)\qquad Q_0(t)\, D(A_1) \subset D(A_1) \quad \textit{and} \quad Q_0(t)\, A_1 x = A_1 Q_0(t)\, x \quad \textit{for} \quad x \in D(A_1),$$

$$(4.7)\qquad Q_{m+1}(t)\, X_1 \subset D(A_1) \quad \textit{and} \quad Q_{m+1}(t)\, A_1 \subset A_1 Q_{m+1}(t)|_{X_1} = \frac{1}{k_m(t)}(Q_m(t) - I)|_{X_1}$$

for all $m \geqslant 0$ and $t > 0$.

Proof. Relation (4.4) follows from (S2). It implies $Q_0(t)\, X_1 \subset X_1$. To show (4.6), let $x \in D(A_1)$. Then $x \in D(A)$, $Ax \in X_1$, and $A_1 x = Ax$. By (4.4) we have $Q_0(t)\, x \in D(A)$ and $AQ_0(t)\, x = Q_0(t)\, Ax = Q_0(t)\, A_1 x \in Q_0(t)\, X_1 \subset X_1$, so that $Q_0(t)\, x \in D(A_1)$ and $A_1 Q_0(t)\, x = AQ_0(t)\, x = Q_0(t)\, A_1 x$.

To show (4,5) for $m \geqslant 0$, write

$$Q_{m+1}(t)\, x = \frac{1}{a_{m+1} * j_n(t)} [a_m * (a * S)](t)\, x = \frac{1}{a_{m+1} * j_n(t)} \int_0^t a_m(t-s)(a * S)(s)\, x\, ds$$

for all $x \in X$. Since the integral

$$\int_0^t Aa_m(t-s)(a * S)(s)\, x\, ds = \int_0^t a_m(t-s)\, A(a * S)(s)\, x\, ds = \int_0^t a_m(t-s)[S(s)\, x - j_n(s)]\, x\, ds$$

exists, the closedness of A implies that $[a_m * (a * S)](t)\, x \in D(A)$ and

$$\begin{aligned} A[a_m * (a * S)](t)\, x &= \int_0^t A a_m(t-s)(a * S)(s)\, x\, ds \\ &= [a_m^* A(a * S)](t)\, x \\ &= a_m^* S(t)\, x - a_m * j_n(t)\, x. \end{aligned}$$

Hence $Q_{m+1}(t)\, x \in D(A)$ and

$$\begin{aligned} AQ_{m+1}(t)\, x &= \frac{1}{a_{m+1} * j_n(t)} [a_m^* A(a * S)](t)\, x \\ &= \frac{1}{a_{m+1} * j_n(t)} [a_m^* S(t)\, x - a_m * j_n(t)\, x] \\ &= \frac{1}{k_m(t)} [Q_m(t)\, x - x] \end{aligned}$$

for all $x \in X$. Moreover, if $x \in D(A)$, then by (S2) and (S3) we have

$$\begin{aligned} AQ_{m+1}(t)\, x &= \frac{1}{a_{m+1} * j_n(t)} [a_m * A(a * S)](t)\, x \\ &= \frac{1}{a_{m+1} * j_n(t)} [a_m * (a * S)](t)\, Ax \\ &= Q_{m+1}(t)\, Ax. \end{aligned}$$

This shows (4.5).

To show (4.7), let $x \in X_1$ and let $\{x_n\} \subset D(A)$ converge to x. $R(Q_{m+1}(t)) \subset D(A)$ implies $Q_{m+1}(t)\, x \in D(A)$. Since A is closed, $AQ_{m+1}(t)$ is bounded, so that $AQ_{m+1}(t)\, x = \lim_{n \to \infty} AQ_{m+1}(t)\, x_n = \lim_{n \to \infty} Q_{m+1}(t)\, Ax_n \in \overline{D(A)} = X_1$. This and (4.5) show that $Q_{m+1}(t)\, x \in D(A_1)$ and $A_1 Q_{m+1}(t)\, x = AQ_{m+1}(t)\, x = \frac{1}{k_m(t)} (Q_m(t) - I)\, x$ for all $x \in X_1$. When $x \in D(A_1)$, we have $x \in D(A)$, $Ax \in X_1$, and $A_1 x = Ax$ so that $Q_{m+1}(t)\, A_1 x = Q_{m+1}(t)\, Ax = AQ_{m+1}(t)\, x = A_1 Q_{m+1}(t)\, x$. This completes the proof.

LEMMA 4.2. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be nondecreasing and positive, and let $S(\cdot)$ be an n-times integrated solution family for (VE, A, a, f) such that $\|S(t)\| \leqslant Mt^n$ for all $t \geqslant 0$.*

(i) *For $m \geqslant 0$, $\|Q_m(t)\, x - x\| \to 0$ as $t \to 0^+$ if and only if $Q_m(t)\, x \to x$ weakly as $t \to 0^+$, if and only if there is a sequence $\{t_n\}$ such that $Q_m(t_n)\, x \to x$ weakly for the case $m \geqslant 1$, if and only if $x \in X_1$.*

(ii) *If $n=0$, then A is densely defined in X.*

Proof. (i) It follows from (4.2), (4.3), and (4.5) that for all $m \geqslant 0$

$$\begin{aligned}\|Q_m(t)\, x - x\| &\leqslant k_m(t)\, \|Q_{m+1}(t)\|\, \|Ax\| \\ &\leqslant k_m(t)\, Mn!\, \|Ax\| \to 0\end{aligned}$$

as $t \to 0^+$ for all $x \in D(A)$, and hence $Q_m(t)\, x \to x$ for all $x \in X_1$, by (4.3). Conversely, from the estimate,

(4.8)
$$\begin{aligned}&|\langle Q_{m+1}(t)\, x - x, x^* \rangle| \\ &\quad = \frac{1}{a_{m+1} * j_n(t)} \left| \left\langle \int_0^t a(t-s)(a_m * S(s)\, x)\, ds \right.\right. \\ &\quad \left.\left. - \int_0^t a(t-s)(a_m * j_n)(s)\, x\, ds, x^* \right\rangle \right| \\ &\quad \leqslant \frac{1}{a_{m+1} * j_n(t)} \int_0^t a(t-s)(a_m * j_n)(s)\, |\langle Q_m(s)\, x - x, x^* \rangle|\, ds \\ &\quad \leqslant \sup\{|\langle Q_m(s)\, x - x, x^* \rangle|;\, 0 \leqslant s \leqslant t\}, \qquad x \in X, \qquad x^* \in X^*,\end{aligned}$$

one sees that if $Q_m(t)\, x \to x$ weakly, then $Q_{m+1}(t)\, x \to x$ weakly, which and the fact that $R(Q_{m+1}(t)) \subset D(A)$ show that $x \in X_1$. When $m \geqslant 1$, $R(Q_m(t_n)) \subset D(A)$, and so $x = \text{w-lim}\, Q_m(t_n)\, x \in X_1$.

(ii) When $n=0$, since $Q_0(t) = S(t) \to I$ strongly as $t \to 0^+$, (4.8) implies that

$$\|Q_1(t)\, x - x\| \leqslant \sup\{\|S(s)\, x - x\|;\, 0 \leqslant s \leqslant t\} \to 0$$

for all $x \in X$. Then we have $X_1 = X$, by the fact that $Q_1(t)\, X \subset D(A)$. That is, A is densely defined for the case $n=0$.

Thus, from (4.4), (4.7) and Lemma 4.2, we see that X_1 is invariant under $Q_m(t)$ for each $m \geqslant 0$, and $\{T_t := Q_m(t)|_{X_1}\}$ is an A_1-regularized approximation process on X_1 with the regularization process $\{S_t := Q_{m+1}(t)|_{X_1}\}$ and with the optimal order $O(k_m(t))(t \to 0^+)$. In particular, $D(A_1)$ is dense in X_1, by Lemma 2.2(ii). Moreover, we have $T_t D(A_1) \subset D(A_1)$ if $m=0$ and $R(T_t) \subset D(A_1)$ if $m \geqslant 1$.

LEMMA 4.3. *The above pair $(\{T_t\}, \{S_t\})$ satisfies* (A2). *If $k_m(t)$ is non-decreasing for t near* 0, *then* (A3) *with $f(t) = (k_m(t))^\beta$ $(0 < \beta \leqslant 1)$ also holds.*

Proof. From (4.8) one can see that $\|S_t - I\|_{X_1} \leqslant \sup\{\|T_s - I\|_{X_1}; 0 \leqslant s \leqslant t\}$, which shows (A2). Moreover, if $\|T_t x - x\| \leqslant M(k_m(t))^\beta$ for all $t \in [0, 1]$, then $\|S_t x - x\| \leqslant M \sup\{(k_m(s))^\beta; 0 \leqslant s \leqslant t\} \leqslant M(k_m(t))^\beta$ for all $t \in [0, 1]$, showing (A3).

From Lemmas 2.2 and 4.3 and Theorem 2.4 we deduce the following uniform convergence theorem.

THEOREM 4.4. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be nondecreasing and positive, and let $S(\cdot)$ be an n-times integrated solution family for (VE, A, a, f) such that $\|S(t)\| \leqslant Mt^n$ for all $t \geqslant 0$.*

(i) *For $m \geqslant 0$, $\|Q_m(t) - I\| \to 0$ as $t \to 0^+$ if and only if $A \in B(X)$. In this case, $\|Q_m(t) - I\| = O(k_m(t))(t \to 0^+)$.*

(ii) *When X_1 is a Grothendieck space with the Dunford–Pettis property, A must be bounded on X, and consequently $\|S(t) - j_n(t)\, I\| = O(a * j_n(t))(t \to 0^+)$.*

Proof. (i) This follows from Lemmas 2.2 and 4.3.

(ii) Applying Theorem 2.4 to $\{T_t := Q_1(t)|_{X_1}\}$ yields that A_1 is bounded on X_1, so that $\|Q_1(t)|_{X_1} - I|_{X_1}\| \leqslant k_1(t)\, \|A_1\|\, \|Q_2(t)\| \leqslant k_1(t)\, \|A_1\|\, Mn! \to 0$ as $t \to 0^+$. Hence $Q_1(t)|_{X_1}$ is invertible on X_1 for small t. Then by (4.5) we have $X_1 = R(Q_1(t)|_{X_1}) \subset R(Q_1(t)) \subset D(A)$, which shows that $D(A)$ is closed and A is bounded. Due to Lemma 4.3, (iii) and (iv) of Lemma 2.2 together imply that $A \in B(X)$. By (i), $\|Q_m(t) - I\| = O(k_m(t))(t \to 0^+)$, and in particular, $\|S(t) - j_n(t)\, I\| = O(a * j_n(t))(t \to 0^+)$.

From Theorems 2.6, 2.7, 2.9, and Lemma 4.3 we can deduce the next theorem.

THEOREM 4.5. *Let $S(\cdot)$ be as assumed in Theorem 4.4 and let $m \geqslant 0$, $0 < \beta \leqslant 1$, and $x \in X_1 = \overline{D(A)}$.*

(i) *$\|Q_m(t)\, x - x\| = o(k_m(t))(t \to 0^+)$ if and only if $x \in N(A_1) = N(A)$.*

(ii) *$\|Q_m(t)\, x - x\| = O(k_m(t))(t \to 0^+)$ if and only if $x \in \widetilde{D(A_1)}^{X_1}$ ($= D(A_1)$, if X is reflexive).*

(iii) *If $K(k_m(t), x, X_1, D(A_1), \|\cdot\|_{D(A_1)}) = O((k_m(t))^\beta)(t \to 0^+)$, then $\|Q_m(t)\, x - x\| = O((k_m(t))^\beta)(t \to 0^+)$. The converse is also true if $k_m(t)$ is nondecreasing for t near 0.*

(iv) *A is unbounded if and only if for some(each) $0 < \beta < 1$ and $m \geqslant 0$ there exist $x^*_{m,\beta} \in X_1 = \overline{D(A)}$ such that*

$$\|Q_m(t)\, x^*_{m,\beta} - x^*_{m,\beta}\| \begin{cases} = O((k_m(t))^\beta) \\ \neq o((k_m(t))^\beta) \end{cases} \qquad (t \to 0^+).$$

If $m=0$, Theorem 4.5 becomes an approximation theorem; if $m \geqslant 1$, it is a local Cesáro ergodic theorem.

Next, we consider the case that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is Laplace transformable, i.e., there is $\omega \geqslant 0$ such that $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)\, dt < \infty$ for all $\lambda > \omega$. Then it is easy to see that $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 4.6. *Suppose $\hat{a}(\lambda) < \infty$ for all $\lambda > \omega$, and let $S(\cdot)$ be an n-times integrated solution family for (VE, A, a, f) such that $\|S(t)\| \leqslant Mt^n$ for all $t \geqslant 0$. Then $(\hat{a}(\lambda))^{-1} \in \rho(A)$, $((\hat{a}(\lambda))^{-1} - A)^{-1} = \lambda^{n+1} \hat{a}(\lambda)\, \hat{S}(\lambda)$, and $\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1}\| \leqslant Mn!$ for all $\lambda > \omega$.*

Proof. Under the assumption (4.1), we can take Laplace transform of the equation in (S3) to obtain

$$\hat{S}(\lambda)\, x = \begin{cases} \dfrac{1}{\lambda^{n+1}}\, x + \hat{a}(\lambda)\, \hat{S}(\lambda)\, Ax, & x \in D(A), \\[2ex] \dfrac{1}{\lambda^{n+1}}\, x + A \hat{a}(\lambda)\, \hat{S}(\lambda)\, x, & x \in X \end{cases}$$

for $\lambda > \omega$. Thus

$$\lambda^{n+1} \hat{a}(\lambda)\, \hat{S}(\lambda)((\hat{a}(\lambda))^{-1} - A) \subset ((\hat{a}(\lambda))^{-1} - A)\, \lambda^{n+1} \hat{a}(\lambda)\, \hat{S}(\lambda) = I,$$

that is, $(\hat{a}(\lambda))^{-1} \in \rho(A)$ and $((\hat{a}(\lambda))^{-1} - A)^{-1} = \lambda^{n+1} \hat{a}(\lambda)\, \hat{S}(\lambda)$ for $\lambda > \omega$. Moreover, (4.1) implies

$$\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1}\| = \|\lambda^{n+1} \hat{S}(\lambda)\| = \left\| \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t)\, dt \right\| \leqslant Mn!.$$

Thus A is a generalized Hille–Yosida operator. Now the following local Abelian ergodic theorem follows immediately from Theorems 3.1, 3.2, and 3.4.

Theorem 4.7. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be nondecreasing, positive, and Laplace transformable, and let $S(\cdot)$ be an n-times integrated solution family for (VE, A, a, f) such that $\|S(t)\| \leqslant Mt^n$ for all $t \geqslant 0$.*

(i) *$\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1} x - x\| \to 0$ as $\lambda \to \infty$ if and only if $x \in X_1$.*

(ii) *$\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| \to 0$ as $\lambda \to \infty$ if and only if $A \in B(X)$. In this case, $\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| = O(\hat{a}(\lambda))(\lambda \to \infty)$.*

(iii) *For $x \in X_1$, $\|(\hat{a}(\lambda))^{-1} ((\hat{a}(\lambda))^{-1} - A)^{-1} x - x\| = o(\hat{a}(\lambda))(\lambda \to \infty)$ if and only if $x \in N(A)$.*

(iv) *For* $0<\beta\leqslant 1$ *and* $x\in X_1$, $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1}-A)^{-1}x-x\| = O((\hat{a}(\lambda))^{\beta})(\lambda\to\infty)$ *if and only if* $K(t, x, X, D(A), \|\cdot\|_{D(A)}) = O(t^{\beta})(t\to 0^+)$, *if and only if* $x\in \widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta=1$, *if and only if* $x\in D(A_1)$ *in the case that* $\beta=1$ *and* X *is reflexive.*

(v) A *is unbounded if and only if for each* $0<\beta<1$ *there exists* $x^*_\beta\in X_1$ *such that*

$$\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1}-A)^{-1}x^*_\beta-x^*_\beta\|\begin{cases} = O((\hat{a}(\lambda))^{\beta}) \\ \neq o((\hat{a}(\lambda))^{\beta})\end{cases} \qquad (\lambda\to\infty).$$

Remarks. (i) Direct proofs for Theorems 4.4(i) and 4.5 have been given in [8] for the case $n=0$, $m=0, 1$.

(ii) Theorem 4.4(ii) implies in particular that every resolvent family $S(\cdot)$ on a Grothendieck space with the Dunford–Pettis property satisfies $\|S(t)-I\| = O(\int_0^t a(s)\,ds)(t\to 0^+)$. Specialization for the cases $a\equiv 1$ and $a(t)=t$ yields the same assertion for C_0-semigroups [19] and cosine operator functions [25].

If one takes $a\equiv 1$, then $k_0(t)=\frac{t}{n+1}$, $k_1(t)=\frac{t}{n+2}$, $Q_0(t)=(n!/t^n)\,S(t)$, and $Q_1(t)=((n+1)!/t^{n+1})\int_0^t S(s)\,ds$. In this case, $S(\cdot)$ becomes an n-times integrated semigroup $T(\cdot)$ with generator A (cf. [1, 17]). Then a combination of applications of Theorems 4.4 and 4.5 to $Q_0(t)$ and $Q_1(t)$ and of Theorem 4.7 leads to the following approximation and local ergodic theorem.

THEOREM 4.8. *Let* $T(\cdot)$ *be an n-times integrated semigroup with generator* A *and satisfying* $\|T(t)\|\leqslant Mt^n$ *for all* $t\geqslant 0$.

(i) $\|(n!/t^n)\,T(t)\,x-x\|\to 0$ *as* $t\to 0^+$ *if and only if* $\|((n+1)!/t^{n+1})\int_0^t T(s)\,x\,ds-x\|\to 0$ *as* $t\to 0^+$, *if and only if* $\|\lambda(\lambda-A)^{-1}x-x\|\to 0$ *as* $\lambda\to\infty$, *if and only if* $x\in X_1$.

(ii) $\|(n!/t^n)\,T(t)-I\|\to 0$ *as* $t\to 0^+$ *if and only if* $\|((n+1)!/t^{n+1})\int_0^t T(s)\,ds-I\|\to 0$ *as* $t\to 0^+$, *if and only if* $\|\lambda(\lambda-A)^{-1}-I\|\to 0$ *as* $\lambda\to\infty$, *if and only if* $A\in B(X)$. *In this case,* $\|(n!/t^n)\,T(t)-I\| = O(t)(t\to 0^+)$ *if and only if* $\|((n+1)!/t^{n+1})\int_0^t T(s)\,ds-I\| = O(t)(t\to 0^+)$, *and* $\|\lambda(\lambda-A)^{-1}-I\| = O(\lambda^{-1})(\lambda\to\infty)$.

(iii) *For* $x\in X_1$, $\|(n!/t^n)\,T(t)\,x-x\| = o(t)(t\to 0^+)$ *if and only if* $\|((n+1)!/t^{n+1})\int_0^t T(s)\,x\,ds-x\| = o(t)(t\to 0^+)$, *if and only if* $\|\lambda(\lambda-A)^{-1}x-x\| = o(\lambda^{-1})(\lambda\to\infty)$, *if and only if* $x\in N(A_1)=N(A)$.

(iv) *For* $0<\beta\leqslant 1$ *and* $x\in X_1$, *the following are equivalent:*

(a) $\|(n!/t^n)\,T(t)\,x-x\| = O(t^{\beta})(t\to 0^+)$;

(b) $\|((n+1)!/t^{n+1})\int_0^t T(s)\,x\,ds-x\| = O(t^{\beta})(t\to 0^+)$;

(c) $\|\lambda(\lambda-A)^{-1}x-x\|=O(\lambda^{-\beta})(\lambda\to\infty)$;

(d) $K(t, x, X, D(A), \|\cdot\|_{D(A)})=O(t^{\beta})(t\to 0^+)$;

(e) $x\in\widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta=1$;

(f) $x\in D(A_1)$ *in the case that* $\beta=1$ *and* X *is reflexive.*

(v) *A is unbounded if and only if for some(each)* $0<\beta<1$ *there exist* $x^*_{1,\beta}, x^*_{2,\beta}, x^*_{3,\beta}\in X_1=\overline{D(A)}$ *such that*

$$\left\|\frac{n!}{t^n}T(t)\,x^*_{1,\beta}-x^*_{1,\beta}\right\|\begin{cases}=O(t^{\beta})\\ \neq o(t^{\beta})\end{cases}\qquad (t\to 0^+),$$

$$\left\|\frac{(n+1)!}{t^{n+1}}\int_0^t T(s)\,x^*_{2,\beta}\,ds-x^*_{2,\beta}\right\|\begin{cases}=O(t^{\beta})\\ \neq o(t^{\beta})\end{cases}\qquad (t\to 0^+),$$

and

$$\|\lambda(\lambda-A)^{-1}x^*_{3,\beta}-x^*_{3,\beta}\|\begin{cases}=O(\lambda^{-\beta})\\ \neq o(\lambda^{-\beta})\end{cases}\qquad (\lambda\to\infty).$$

If one takes $a(t)=t$, then $k_0(t)=t^2/(n+1)(n+2)$, $k_1(t)=t^2/(n+3)(n+4)$, $Q_0(t)=(n!/t^n)\,S(t)$, and $Q_1(t)=((n+2)!/t^{n+2})\int_0^t(t-s)\,S(s)\,ds$. In this case, $S(\cdot)$ becomes an n-times integrated semigroup $C(\cdot)$ with generator A (cf. [29]). Similarly, one can deduce from Theorems 4.4, 4.5, and 4.7 the following approximation and local ergodic theorem (cf. [9]).

THEOREM 4.9. *Let* $C(\cdot)$ *be an n-times integrated cosine function with generator A and satisfying* $\|C(t)\|\leqslant Mt^n$ *for all* $t\geqslant 0$.

(i) $\|(n!/t^n)\,C(t)\,x-x\|\to 0$ *as* $t\to 0^+$ *if and only if* $\|((n+2)!/t^{n+2})\int_0^t(t-s)\,C(s)\,x\,ds-x\|\to 0$ *as* $t\to 0^+$, *if and only if* $\|\lambda(\lambda-A)^{-1}x-x\|\to 0$ *as* $\lambda\to\infty$, *if and only if* $x\in X_1$.

(ii) $\|(n!/t^n)\,C(t)-I\|\to 0$ *as* $t\to 0^+$ *if and only if* $\|((n+2)!/t^{n+2})\int_0^t(t-s)\,C(s)\,ds-I\|\to 0$ *as* $t\to 0^+$, *if and only if* $\|\lambda(\lambda-A)^{-1}-I\|\to 0$ *as* $\lambda\to\infty$, *if and only if* $A\in B(X)$. *In this case,* $\|(n!/t^n)\,C(t)-I\|=O(t)$ $(t\to 0^+)$ *if and only if* $\|((n+2)!/t^{n+2})\int_0^t(t-s)\,C(s)\,ds-I\|=O(t)(t\to 0^+)$, *and* $\|\lambda(\lambda-A)^{-1}-I\|=O(\lambda^{-1})(\lambda\to\infty)$.

(iii) *For* $x\in X_1$, $\|(n!/t^n)\,C(t)\,x-x\|=o(t^2)(t\to 0^+)$ *if and only if* $\|((n+2)!/t^{n+2})\int_0^t(t-s)\,C(s)\,x\,ds-x\|=o(t^2)(t\to 0^+)$, *if and only if* $\|\lambda(\lambda-A)^{-1}x-x\|=o(\lambda^{-1})(\lambda\to\infty)$, *if and only if* $x\in N(A_1)=N(A)$.

(iv) *For* $0<\beta\leqslant 1$ *and* $x\in X_1$, *the following are equivalent:*

(a) $\|(n!/t^n)\,C(t)\,x-x\|=O(t^{2\beta})(t\to 0^+)$;

(b) $\|((n+2)!/t^{n+2})\int_0^t(t-s)\,C(s)\,x\,ds-x\|=O(t^{2\beta})(t\to 0^+)$;

(c) $\|\lambda(\lambda-A)^{-1}x-x\|=O(\lambda^{-\beta})(\lambda\to\infty)$;

(d) $K(t, x, X, D(A), \|\cdot\|_{D(A)})=O(t^{2\beta})(t\to 0^+)$;

(e) $x\in\widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta=1$;

(f) $x\in D(A_1)$ *in the case that* $\beta=1$ *and* X *is reflexive.*

(v) *A is unbounded if and only if for some(each)* $0<\beta<1$ *there exist* $x^*_{1,\beta}, x^*_{2,\beta}, x^*_{3,\beta}\in X_1=\overline{D(A)}$ *such that*

$$\left\|\frac{n!}{t^n}C(t)\,x^*_{1,\beta}-x^*_{1,\beta}\right\|\begin{cases}=O(t^{2\beta})\\ \neq o(t^{2\beta})\end{cases}\qquad(t\to 0^+),$$

$$\left\|\frac{(n+2)!}{t^{n+2}}\int_0^t(t-s)\,C(s)\,x^*_{2,\beta}\,ds-x^*_{2,\beta}\right\|\begin{cases}=O(t^{2\beta})\\ \neq o(t^{2\beta})\end{cases}\qquad(t\to 0^+),$$

and

$$\|\lambda(\lambda-A)^{-1}\,x^*_{3,\beta}-x^*_{3,\beta}\|\begin{cases}=O(\lambda^{-\beta})\\ \neq o(\lambda^{-\beta})\end{cases}\qquad(\lambda\to\infty).$$

Remark. Theorems 4.8 and 4.9, except assertion (ii), were originally proved in [9]. When $n=0$, these theorems further reduce to some results in [3–7].

5. APPROXIMATION OF (Y)-SEMIGROUPS AND TENSOR PRODUCT SEMIGROUPS

In this section, we apply the results in Section 2 to (Y)-*semigroups.* Let Y be a closed subspace of X^* such that the canonical embedding of X into Y^* is isometric. A semigroup $\{T(t);t\geqslant 0\}$ of operators on X is called a (Y)-semigroup [24] if Y is invariant under $T^*(t)$ for all $t\geqslant 0$ and $T(\cdot)\,x$ is $\sigma(X,Y)$-continuous and locally $\sigma(X,Y)$-Pettis integrable on $[0,\infty)$ for each $x\in X$. The Y-generator A of $T(\cdot)$ is defined by $Ax:=\sigma(X,Y)\text{-}\lim_{t\to 0^+}t^{-1}(T(t)-I)\,x$.

THEOREM 5.1. *Let* $T(\cdot)$ *be a* (Y)-*semigroup which is uniformly bounded on* $[0,1]$. *We have:*

(i) $\|T(t)-I\|\to 0$ *if and only if* $A\in B(X)$. *In this case, we have* $\|T(t)-I\|=O(t)(t\to 0^+)$. *This always holds if* X_1 *is a Grothendieck space with the Dunford–Pettis property.*

(ii) $\|T(t)\,x-x\|\to 0$ *as* $t\to 0^+$ *if and only if* $x\in\overline{D(A)}$.

(iii) $\|T(t)\,x-x\|=o(t)$ $(t\to 0^+)$ *if and only if* $x\in N(A)$.

(iv) *For* $0<\beta\leqslant 1$ *and* $x\in X_1=\overline{D(A)}$, *one has* $\|T(t)\,x-x\|=O(t^{\beta})$ $(t\to 0^+)$ *if and only if* $K(t,x,X_1,D(A_1),\|\cdot\|_{D(A_1)})=O(t^{\beta})(t\to 0^+)$, *if and only if* $x\in \widetilde{D(A_1)}^{X_1}$ *in the case* $\beta=1$.

(v) *A is unbounded if and only if for every (some)* $0<\beta<1$ *there is* $x_\beta\in\overline{D(A)}$ *such that*

$$\|T(t)\,x_\beta-x_\beta\|\begin{cases}=O(t^{\beta})\\ \neq o(t^{\beta})\end{cases}\qquad (t\to 0^+).$$

Proof. Let $S(t)\,x:=\sigma(X,Y)\text{-}\int_0^t T(s)\,x\,ds$, $x\in X$, $t\geqslant 0$. $T(t)$ is a bounded linear operator on X, $T(\cdot)$ is uniformly bounded in every bounded closed subinterval $[a,b]$ of $(0,\infty)$, and $S(\cdot)$ is continuous on $(0,\infty)$ in operator norm [24, Lemma 2.1]. Moreover, if $T(\cdot)$ is uniformly bounded on $[0,1]$, then $\|S(t)\|=O(t)(t\to 0^+)$. We also know [24, Proposition 2.4] that

$$T(t)\,D(A)\subset D(A)\quad\text{and}\quad T(t)\,Ax=AT(t)\,x\quad\text{for } x\in D(A),\ t>0,\tag{5.1}$$

$$R(S(t))\subset D(A)\quad\text{and}\quad S(t)\,A\subset AS(t)=T(t)-I\quad\text{for } t>0.\tag{5.2}$$

It is easy to show that $T(t)\,x\to x$ implies $S(t)\,x\to x$ as $t\to 0^+$. If $T(\cdot)$ is uniformly bounded on $[0,1]$, then we see from (5.2) that $T(t)\,x\to x$ if and only if $x\in\overline{D(A)}$. From (5.1) and (5.2) we also see that $X_1=\overline{D(A)}$ is invariant under $T(t)$ and $S(t)$, and $\{T_t:=T(t)|_{X_1}\}_{t>0}$ is an A_1-regularized approximation process on X_1 of order $O(t)(t\to 0^+)$, with the regularization process $\{S_t:=t^{-1}S(t)|_{X_1}\}_{t>0}$. By taking integrals, one can see that $\{T_t:=t^{-1}S(t)|_{X_1}\}_{t>0}$ is an A_1-regularized approximation process on X_1 with the regularization process $\{S_t:=2t^{-2}\int_0^t S(s)\,ds|_{X_1}\}_{t>0}$. One can also check (A2) and (A3) with $f=t^{\beta}(0<\beta\leqslant 1)$. Hence the assertions can be deduced from Lemma 2.2, Theorems 2.4, 2.6, 2.7, and 2.9.

In particular, a C_0-semigroup on X is a (Y)-semigroup on X^* with $Y=X^*$, and its dual semigroup is a (Y)-semigroup on X^* with $Y=X\subset X^{**}$. Another example of (Y)-semigroup is the tensor product semigroup of two C_0-semigroups.

For $i=1,2$, let X_i be a Banach space and $\{T_i(t);t\geqslant 0\}\subset B(X_i)$ be a (C_0)-semigroup with the infinitesimal generator A_i. Suppose $\|T_i(t)\|\leqslant M_i e^{w_i t}$, $t\geqslant 0$, $i=1,2$. The family $\{\mathrm{S}(t);t\geqslant 0\}$ of operators on $B(X_2,X_1)$, defined by $\mathrm{S}(t)\,E=T_1(t)\,ET_2(t)(E\in B(X_2,X_1))$, is a semigroup in the algebra $B(B(X_2, X_1))$, and is called the tensor product semigroup of $T_1(\cdot)$ and $T_2(\cdot)$ (see [14, 23]). Let Y be the closed linear span of the set $\{f_{x_2, x_1^*}; x_2 \in X_2, x_1^* \in X_1^*\}$, where f_{x_2, x_1^*} is the linear functional on $B(X_2, X_1)$ defined by $\langle B, f_{x_2, x_1^*} \rangle = \langle Bx_2, x_1^* \rangle$.

LEMMA 5.2 [24, 30]. *$S(\cdot)$ is a (Y)-semigroup on $B(X_2, X_1)$, and the (Y)-generator Δ of $S(\cdot)$, which is defined by $\Delta E := $ so-$\lim_{t \to 0^+} t^{-1}(S(t)E - E)$, is precisely the operator which has as its domain*

$$D(\Delta) = \{E \in B(X_2, X_1); ED(A_2) \subset D(A_1) \text{ and } A_1 E + EA_2 \text{ is bounded on } D(A_2)\},$$

and sends each such $E \in D(\Delta)$ to $\overline{A_1 E + EA_2}$.

Δ is closed relative to the weak operator topology and densely defined relative to the strong operator topology (see [30, Proposition 3.3]). For $\lambda > w_1 + w_2$, $\lambda - \Delta$ is invertible and

$$(\lambda - \Delta)^{-1} Ex = \int_0^\infty e^{-\lambda t}(S(t)E)x\,dt \qquad (E \in B(X_2, X_1), x \in X_2).$$

If $w_1 + w_2 \leqslant 0$, then $\|S(t)\| \leqslant M_1 M_2$ for all $t \geqslant 0$, so that Theorem 5.1 ((ii)–(v)) can apply to $S(\cdot)$. Also we have $(0, \infty) \subset \rho(\Delta)$ and $\|\lambda(\lambda - \Delta)^{-1}\| \leqslant M_1 M_2$ for all $\lambda > 0$. Hence the operator Δ is a generalized Hille–Yosida operator of type $(M_1 M_2, 0)$ on $B(B(X_2, X_1))$, so that the results in Section 3 can be applied to Δ. Thus we can formulate the following theorem.

THEOREM 5.3. *Suppose that $w_1 + w_2 \leqslant 0$. We have:*

(i) *$\|T_1(t)ET_2(t) - E\| \to 0$ as $t \to 0^+$ if and only if $\|\lambda(\lambda - \Delta)^{-1}E - E\| \to 0$ as $\lambda \to \infty$, if and only if $E \in \overline{D(\Delta)}$.*

(ii) *$\|T_1(t)ET_2(t) - E\| = o(t)$ $(t \to 0^+)$ if and only if $\|\lambda(\lambda - \Delta)^{-1}E - E\| = o(\lambda^{-1})(\lambda \to \infty)$, if and only if $E \in N(\Delta)$.*

(iii) *For $0 < \beta \leqslant 1$ and $E \in \overline{D(\Delta)}$, one has $\|T_1(t)ET_2(t) - E\| = O(t^\beta)$ $(t \to 0^+)$ if and only if $\|\lambda(\lambda - \Delta)^{-1}E - E\| = O(\lambda^{-\beta})(\lambda \to \infty)$, if and only if $K(t, E, B(X_2, X_1), D(\Delta), \|\cdot\|_{D(\Delta)}) = O(t^\beta)$ $(t \to 0^+)$. Moreover, in case $\beta = 1$, these conditions are equivalent to that $E \in \widetilde{D(\Delta)}^{B(X_2, X_1)}$.*

(iv) *Δ is unbounded if and only if for every (some) $0 < \beta < 1$ there are $F_\beta, F'_\beta \in \overline{D(\Delta)}$ such that*

$$\|T_1(t)F_\beta T_2(t) - F_\beta\| \begin{cases} = O(t^\beta) \\ \neq o(t^\beta) \end{cases} \qquad (t \to 0^+),$$

and

$$\|\lambda(\lambda-\Delta)^{-1} F'_\beta - F'_\beta\| \begin{cases} = O(\lambda^{-\beta}) \\ \neq o(\lambda^{-\beta}) \end{cases} \qquad (\lambda \to \infty).$$

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